§ 5.5 Higher Derivatives

One can define higher derivatives iteratively.
Let $m \in \mathbb{N}$.

Definition 5.3:

i) f is called "m-times differentiable" on Ω, if f is $(m-1)$-times differentiable with $(m-1)$-th derivative $f^{(m-1)}$ differentiable.
In this case

$$f^{(m)} = \frac{df^{(m-1)}}{dx} = \frac{d^m f}{dx^m} : \Omega \longrightarrow \mathbb{R}$$

is called the "m-th derivative" of f.
Note: $f^{(0)}$ is just the function f itself.

ii) f is of the class $\mathcal{C}^m(\Omega)$, if f is m-times differentiable and if the functions $f = f^{(0)}$, $f' = f^{(1)}, \ldots, f^{(m)}$ are continuous.

Notation:

$$\mathcal{C}^m(\Omega, \mathbb{R}) = \{ f : \Omega \to \mathbb{R} \mid f \text{ is } m\text{-times diff.}, \; f, \ldots, f^{(m)} \text{ continuous} \}$$

Example 5.15:

The functions exp, sin, cos, polynomials and rational functions are in $\mathcal{C}^m$ for each $m \in \mathbb{N}$.

Let now $\Omega = (a,b)$, $-\infty < a < b < \infty$, and $m \in \mathbb{N}$.

Proposition 5.11 (Taylor-formula):

Let $f \in C^{m-1}([a,b])$ on (a,b) m-times differentiable. Then there exists $\xi \in (a,b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + f''(a)\frac{(b-a)^2}{2} + \dots$$

$$\dots + f^{(m-1)}(a)\frac{(b-a)^{m-1}}{(m-1)!} + f^{(m)}(\xi)\frac{(b-a)^m}{m!}$$

Proof:

We can trace the Proposition to the mean value theorem, namely Prop. 5.9.

Consider the function

$$g(x) = f(x) + f'(x)(b-x) + \dots$$

$$(*) \qquad + f^{(m-1)}(x)\frac{(b-x)^{m-1}}{(m-1)!} + K\frac{(b-x)^m}{m!} - f(b),$$

where $K \in \mathbb{R}$ is chosen such that $g(a) = g(b) = 0$.
According to assumptions on f, g is continuous on $[a,b]$, and differentiable on (a,b).
Then Prop. 5.9 $\Longrightarrow \exists\, \xi \in (a,b)$: $g'(\xi) = 0$
That is

$$
\begin{aligned}
0 = \; & f'(\xi) + \left(f''(\xi)(b-\xi) - f'(\xi) \right) \\
& + \left(f'''(\xi) \frac{(b-\xi)^2}{2} - f''(\xi)(b-\xi) \right) + \cdots \\
& + \left(f^{(m)}(\xi) \frac{(b-\xi)^{m-1}}{(m-1)!} - f^{(m-1)}(\xi) \frac{(b-\xi)^{m-2}}{(m-2)!} \right) \\
& - K \frac{(b-\xi)^{m-1}}{(m-1)!} \\
= \; & \left(f^{(m)}(\xi) - K \right) \frac{(b-\xi)^{m-1}}{(m-1)!} ,
\end{aligned}
$$

as all other terms cancel pairwise. As $b - \xi > 0$, we get $K = f^{(m)}(\xi)$, and with $g(a) = 0$ the claim follows after setting $x = a$ in $(*)$.

$\square$

<u>Remark 5.5:</u>
The tangent at $f \in C^1([a,b])$ in the point $x=a$,

$$T_1 f(x;a) = f(a) + f'(a)(x-a),$$

the approximating parabola for $f \in \mathcal{C}^2([a,b])$,

$$T_2 f(x;a) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2},$$

and more generally the taylor polynomial of m-th degree for $f \in \mathcal{C}^m([a,b])$,

$$T_m f(x;a) = f(a) + f'(a)(x-a) + \cdots + f^{(m)}(a)\frac{(x-a)^m}{m!},$$

have according to Prop. 5.11 the following approximating property. For $a<x<b$ we have

$$f(x) - T_m f(x;a) = \left(f^{(m)}(\xi) - f^{(m)}(a)\right) \cdot \frac{(x-a)^m}{m!}$$
$$=: r_m f(x;a)$$

for some $\xi \in (a,b)$. For the remainder term $r_m f$ we obtain the following

$$|r_m f(x;a)| \leq \sup_{a<\xi<x} \left|f^{(m)}(\xi) - f^{(m)}(a)\right| \frac{(x-a)^m}{m!}$$

If $f \in C^{m+1}$, we can improve this through Prop. 5.9:

$$|r_m f(x;a)| \leq \sup_{a<\xi<x} \left|f^{(m+1)}(\xi)\right| \frac{(x-a)^{m+1}}{m!}$$

Example 5.14:

What is the sine of $47° \cong \frac{\pi}{4} + \frac{2\pi}{180}$?

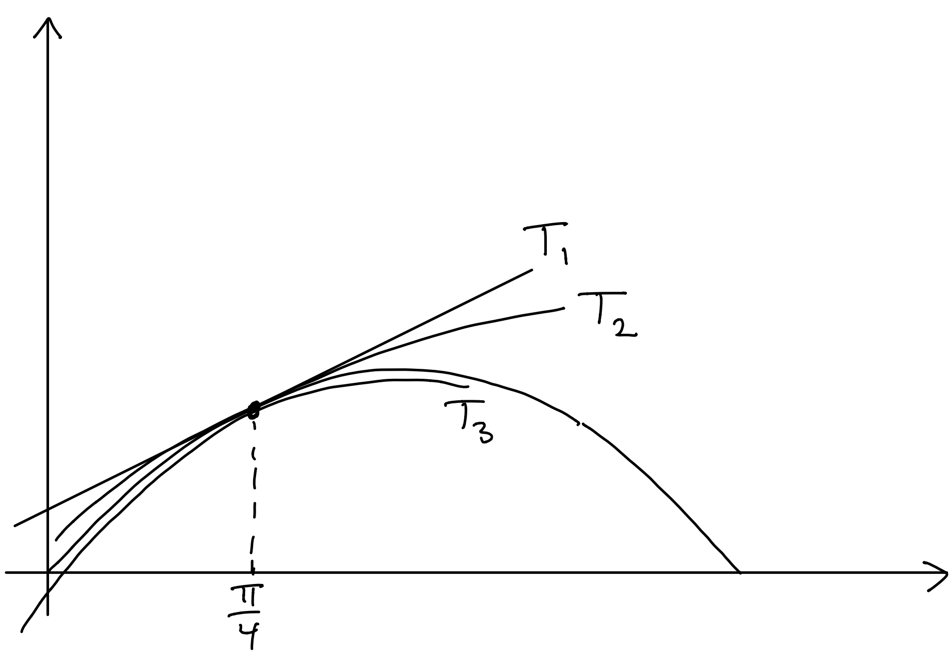
Using $\sin' = \cos$, $\sin'' = \cos' = -\sin$, etc., we obtain from the above remark when choosing $m = 2$ the approximation:

$$\sin\left(\frac{\pi}{4} + \frac{\pi}{90}\right) = \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)\frac{\pi}{90} - \sin\left(\frac{\pi}{4}\right)\frac{\pi^2}{90^2} + r_2$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,\frac{\pi}{90} - \frac{\sqrt{2}}{2}\cdot\frac{\pi^2}{2\cdot 90^2} + r_2,$$

where

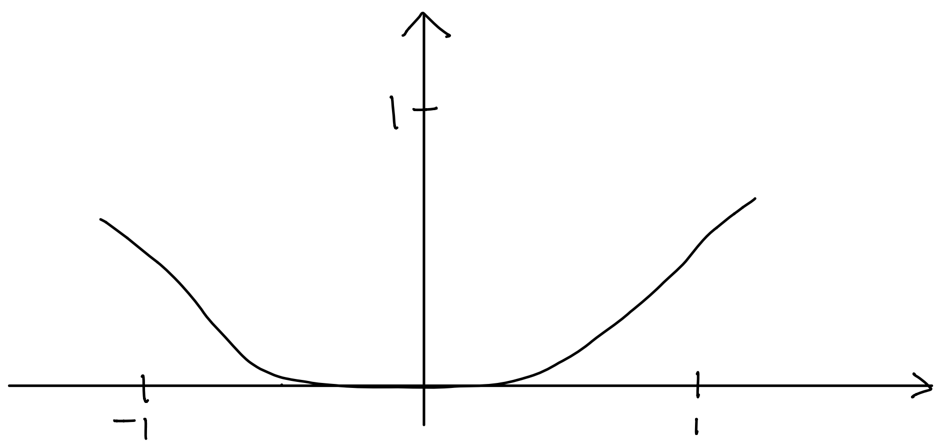
$$|r_2| \leq \frac{\pi^3}{2\cdot 90^3} \approx 10^{-5}.$$



Remark 5.6:

The Taylor-series of a function f does not necessarily converge against f.
Consider for example the function $f: \mathbb{R} \to \mathbb{R}$

$$f(x) := \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$



This function comes very close to the x-axis even for non-zero x:

$$0 < f(x) < 10^{-10} \quad \text{for} \quad 0 < |x| \leq 0{,}2$$

In fact, we have for the n-th derivative:

$$f^{(n)}(x) = \begin{cases} p_n\left(\frac{1}{x}\right) e^{-1/x^2}, & \text{if } x \neq 0 \\ 0 & , \text{if } x = 0 \end{cases}$$

where p_n is some polynomial.

Proof:

Use induction:

- $\underline{n=0}$: clear $(p_0 = 1)$
- $\underline{n \to n+1}$:

i) for $x \neq 0$ we have

$$f^{(n+1)}(x) = \frac{d}{dx} f^{(n)}(x) = \frac{d}{dx}\left(p_n\left(\tfrac{1}{x}\right) e^{-1/x^2}\right)$$

$$= \left(-p_n'\left(\frac{1}{x}\right)\frac{1}{x^2} + 2p_n\left(\frac{1}{x}\right)\frac{1}{x^3}\right) e^{-1/x^2}$$

Choose $p_{n+1}(t) := -p_n'(t)t^2 + 2p_n(t)t^3$.

ii) For $x = 0$ we have

$$f^{(n+1)}(0) = \lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x} = \lim_{x\to 0} \frac{p_n\left(\frac{1}{x}\right) e^{-1/x^2}}{x}$$

$$= \lim_{R \to \pm\infty} R p_n(R) e^{-R^2} = 0$$

$\square$

Thus we see that the Taylor-series of f at $x_0 = 0$ is identical to zero although $f = 0$ only at $x = 0$.

Example 5.15 (logarithm):

For $0 \leq x \leq 1$ the Taylor-series of $\log(1+x)$ converges and we have:

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \mp \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

Proof:

$$\log'(1+x)\Big|_{x=0} = \frac{1}{1+x}\Big|_{x=0} = 1$$

$$\log''(1+x)\Big|_{x=0} = \frac{d}{dx}\left(\frac{1}{1+x}\right)\Big|_{x=0} = -\frac{1}{(1+x)^2}\Big|_{x=0} = -1$$

$\vdots$

$$\log^{(n)}(1+x)\Big|_{x=0} = (-1)^{n-1}(n-1)!\,\frac{1}{(1+x)^n}\Big|_{x=0} = (-1)^{n-1}(n-1)!$$

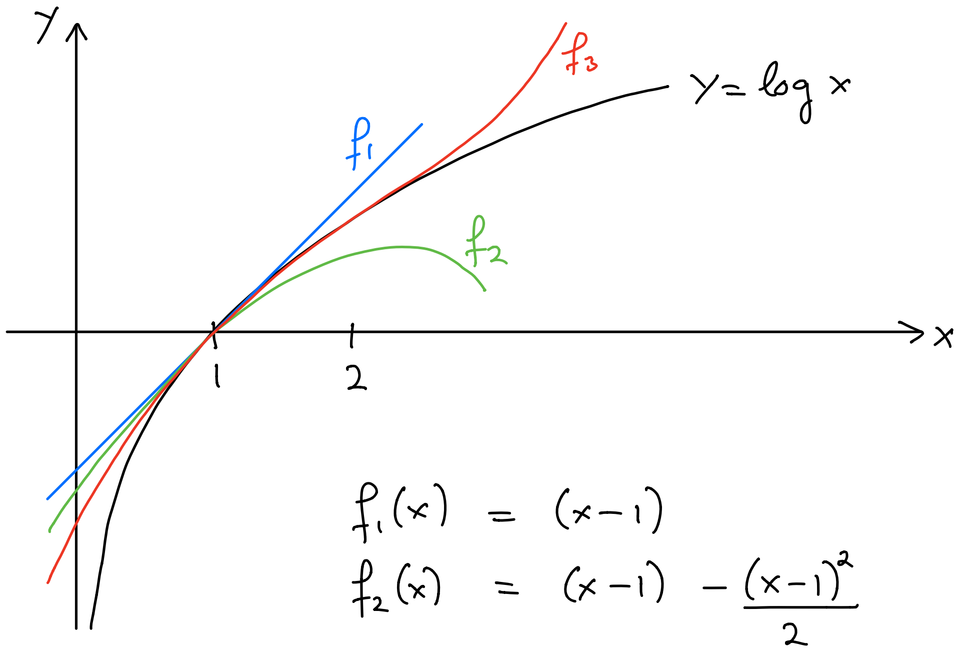
Prop. 5.11 $\Rightarrow$ $\log(1+x)$

$$= \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n} x^n + \underbrace{\log^{(N+1)}(\xi)}_{= \frac{(-1)^N N!}{(1+\xi)^{N+1}}} \frac{x^{N+1}}{(N+1)!}$$

$$= \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n} x^n + \underbrace{\frac{(-1)^N}{N+1} \cdot \frac{x^{N+1}}{(1+\xi)^{N+1}}}_{= r_N \log(1+x;0)}$$

for some $\xi \in (1, x)$

But $|r_N \log(1+x;0)| \leq \frac{1}{N+1} \longrightarrow 0 \quad (N \to \infty)$



$$f_1(x) = (x-1)$$

$$f_2(x) = (x-1) - \frac{(x-1)^2}{2}$$

$$f_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$$

In fact, one can show that the above series is convergent for $|x|<1$. Then one writes

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \pm + \cdots$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

subtraction gives

$$\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) = 2\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$$

Using $\left.\frac{1+x}{1-x}\right|_{x=\frac{1}{3}} = 2$, we then get

$$\log 2 = 2\left(\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \cdots\right)$$

$\longrightarrow$ gives an effective way to compute $\log 2$.

Let $\Omega \subset \mathbb{R}$ be open, $f : \Omega \longrightarrow \mathbb{R}$.

Definition 5.4:

A point $x_0 \in \Omega$ is called a (strict) "local minimum" of f, if in a neighborhood U of x_0 we have

$$\forall\, x \in U : \ f(x) \geq f(x_0),$$
$$\left(\text{or } \forall\, x \in U \setminus \{x_0\} : f(x) > f(x_0)\right).$$

If f is differentiable at a local minimum x_0, then we have following the proof of Prop. 5.9

$$0 \leq \lim_{x \downarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) = \lim_{x \uparrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \leq 0$$

$\Rightarrow f'(x_0) = 0$. More generally, we have the following

Corollary 5.3:

Let $f \in C^m(\Omega)$, $x_0 \in \Omega$ with $f'(x_0) = \dots = f^{(m-1)}(x_0) = 0$

i) If $m = 2K+1$, x_0 local minimum, then $f^{(m)}(x_0) = 0$

ii) If $m = 2K$, and if $f^{(m)}(x_0) > 0$, then x_0 is strict local minimum.

Proof:

According to Prop. 5.11 there exists for $x \in \Omega$ close to x_0 a ξ between x and x_0 such that

$$f(x) = f(x_0) + f^{(m)}(\xi) \frac{(x-x_0)^m}{m!} .$$

i) If $m = 2K+1$, and if x_0 is local minimum, then we get

$$f^{(m)}(x_0) = \lim_{\xi \to x_0} f^{(m)}(\xi) = \begin{cases} m! \lim\limits_{x \downarrow x_0} \frac{f(x)-f(x_0)}{(x-x_0)^m} \geq 0 \\ m! \lim\limits_{x \uparrow x_0} \frac{f(x)-f(x_0)}{(x-x_0)^m} \leq 0 \end{cases}$$

$$\Rightarrow f^{(m)}(x_0) = 0 .$$

ii) For $m = 2K$, $x \neq x_0$ we have $(x-x_0)^m > 0$.
If $f^{(m)}(x_0) = \lim_{\xi \to x_0} f^{(m)}(\xi) > 0$, then $f(x) - f(x_0)$ for $x \neq x_0$ close to x_0; thus x_0 is strict local minimum.

Example 5.16:

i) Let $f(x) = x^4 - x^2 + 1$, $x \in \mathbb{R}$. According Corollary 5.3 i) for having an extremum at x_0, we need
$$f'(x_0) = 4x_0^3 - 2x_0 = 2(2x_0^2 - 1)x_0 = 0 ;$$

that is
$$x_0 \in \left\{ -\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}} \right\}.$$

According to Corollary 5.3 ii) and with
$$f''(x) = 12x^2 - 2 = \begin{cases} 4 > 0, & x = \pm 1/\sqrt{2} \\ -2 < 0, & x = 0 \end{cases}$$

we have strict local minima at $x_0 = \frac{1}{\sqrt{2}}$ and at $x_0 = -\frac{1}{\sqrt{2}}$, and a strict local maximum at $x_0 = 0$.

ii) Let $a_1, \dots, a_n \in \mathbb{R}$. We want to find the "least square"-approximation $x_0 \in \mathbb{R}$ of $(a_i)_{1 \le i \le n}$ with
$$f(x_0) = \sum_{k=1}^{n} (x_0 - a_k)^2 = \min_x f(x).$$

Notice $f(x) \longrightarrow \infty$ $(|x| \to \infty)$; thus there exists a $x_0 \in \mathbb{R}$ with $f(x_0) = \min_x f(x)$.

Corollary 5.3 i) gives the necessary condition: $f'(x_0) = 2\sum_{k=1}^{n}(x_0 - a_k) = 2nx_0 - 2\sum_{k=1}^{n} a_k = 0$

$$\Rightarrow x_0 = \frac{1}{n}\sum_{k=1}^{n} a_k.$$